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On the emptiness formation probability of the open XXZ spin-$1/2$ chain

K. K. Kozlowski[1]

Abstract

This paper is devoted to the study of the emptiness formation probability $\tau(m)$ of the open XXZ chain. We derive a closed form for $\tau(m)$ at $\Delta = 1/2$ when the boundary field vanishes. Moreover we obtain its leading asymptotics for an arbitrary boundary field at the free fermion point. Finally, we compute the first term of the asymptotics of $\ln(\tau(m))$ in the whole massless regime $-1 < \Delta < 1$.

[1] Laboratoire de Physique, ENS Lyon et CNRS, France, karol.kozlowski@ens-lyon.fr

1 Introduction

For already a few decades, spin chains trigger a lot of interest as many relevant physical quantities can be computed. For instance the spectrum ([1], [2]) of the spin-1/2 XXZ chain as well as its correlation functions ([3], [4]) are known. The open XXZ spin-1/2 chain with diagonal boundary conditions:

$$\mathcal{H} = \sum_{m=1}^{M-1} \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left(\sigma_m^z \sigma_{m+1}^z - 1 \right) + \sigma_1^z \, h_- + \sigma_M^z \, h_+, \tag{1.1}$$

is another example of an integrable spin chain Its spectrum was first obtained by Alcaraz et *al.* [5] in the framework of coordinate Bethe ansatz. One year later, Sklyanin [6] developed an appropriate scheme to apply the algebraic Bethe ansatz to models with boundaries. In the mid nineties, the Kyoto group obtained an integral representation for the elementary blocks of a half-infinite massive ($\Delta > 1$) XXZ spin-1/2 chain with a diagonal boundary field [7]. Recently, the Lyon group derived formulas for the elementary blocks of the finite open chain XXZ with diagonal boundary fields and for an arbitrary anisotropy parameter [8].

The simplest possible correlation function of a spin-1/2 chain is the so-called emptiness formation probability (EFP) $\tau(m)$ first introduced in [9]. This quantity can be understood as the probability of observing a ferromagnetic string of length m starting from the first site of the chain. The EFP can be computed to the end in the case of a periodic XXZ spin-1/2 chain at $\Delta = 1/2$. Indeed this exact expression has been conjectured in [10] and proven in [11]. The free fermion point of the bulk model is also quite particular as the EFP can be represented as a Toeplitz determinant of a smooth function on an arc. Hence its asymptotics can be studied [12] by using the strong Szegö limit theorem for circular arcs [13]. One can also derive the leading asymptotics of $\ln \tau(m)$ of the bulk massless model using a saddle point method [14].

The aim of this paper is to obtain corresponding results for the open XXZ spin 1/2 chain. The symmetrized integral representation for the EFP derived in [15] will be the starting point of our considerations. We first show that $\tau(m)$ can be computed to the end in the case of an open spin-1/2 chains subject to a zero boundary field at $\Delta = 1/2$. After setting some notations in Section 2, we will present this result in Section 3. We then derive the leading asymptotics of the EFP at the free fermion point in Section 4. This is done by exploiting the almost Hankel determinant structure of the EFP at $\Delta = 0$. We analyse the asymptotic behavior of such determinants in Appendix A. This analysis takes advantage of the asymptotics of Toeplitz matrices with Fischer-Hartwig type singularities [16] as well as of the uniform asymptotics of orthogonal polynomials of modified Jacobi type [17]. In the last section we use the saddle point approximation borrowed from the bulk [14] to derive the leading asymptotics of $\ln \tau(m)$ in the massless regime.

2 The Emptiness formation probability.

The authors of [15] showed that, in the case of a half-infinite chain, the emptiness formation probability admits a multiple integral representation whose integrand is a symmetric function of the integration variables. This way of representing $\tau(m)$ enables the separation of variables in the m-fold integral defining $\tau(m)$ at $\Delta = 1/2$. It also allows to express $\tau(m)$ as an almost Hankel determinant at $\Delta = 0$. Finally, it is suited for the derivation the leading asymptotics of $\ln[\tau(m)]$ in the massless regime.

We recall the standard parameterizations of the massless regime

$$\Delta = \cos(\zeta) \ , \ \zeta \in]\,0\,;\pi\,[\qquad \text{and} \qquad h_- = -\sin\zeta\cot\hat{\xi}_- \ , \ \hat{\xi}_- \in]\,-\pi/2\,;\pi/2\,]$$

as well as the symmetrized integral representation for the emptiness formation probability [15]:

$$\begin{aligned}\tau(m) &= \langle E_{22}^{(1)}\dots E_{22}^{(m)}\rangle \\ &= \frac{1}{2^m\,m!}\int\limits_{\mathcal{C}(h)} \mathrm{d}^m\lambda \frac{\det_m[\Phi(\lambda_j,\xi_k)]\,\det_m[1/h(\lambda_j,\xi_k)]}{\prod\limits_{i<j}\mathfrak{s}^2(\xi_i,\xi_j)} \\ &\quad\times\frac{\prod\limits_{i,j}^{m} h(\lambda_i,\xi_j)}{\prod\limits_{j>k}^{m}\mathfrak{s}(\lambda_{jk},i\zeta)\,\mathfrak{s}(\overline{\lambda}_{jk},i\zeta)}\prod_{k=1}^{m}\frac{\sinh(2\lambda_k)\,\mathfrak{s}\left(\xi_k,i\hat{\xi}_-\right)}{\mathfrak{s}\left(\lambda_k,i\hat{\xi}_-+i\zeta/2\right)}\end{aligned} \tag{2.1}$$

The contours of this multiple integral depend on the boundary magnetic field h_-: $\mathcal{C}(h) = \mathbb{R}\bigcup_{\pm}\Gamma_{\pm}\left\{\mp i\left(\hat{\xi}_- + \zeta/2\right)\right\}$ whenever $-\zeta/2 < \hat{\xi}_- < 0$ and $\mathcal{C}(h) = \mathbb{R}$ otherwise.

Here $\Gamma_{\pm}$ stands for a small circle skimmed through in the anticlockwise/clockwise direction. Furthermore, we agree that

$$\begin{aligned}&\mathfrak{s}(x,y) = \sinh(x+y)\sinh(x-y) &\qquad& \mathfrak{c}(x,y) = \cosh(x+y)\cosh(x-y) \\ &\lambda_{ij} = \lambda_i - \lambda_j && \overline{\lambda}_{ij} = \lambda_i + \lambda_j\end{aligned} \tag{2.2}$$

and introduce the functions:

$$\Phi(\lambda,\xi) = \frac{\rho(\lambda-\xi)-\rho(\lambda+\xi)}{\sinh(2\xi)} = \frac{\sinh(\pi\lambda/\zeta)\sinh(\pi\xi/\zeta)}{\zeta\sinh(2\xi)\,\mathfrak{c}\left[\frac{\pi}{\zeta}(\lambda,\xi)\right]} \tag{2.3}$$

$$h(\lambda,\xi) = \mathfrak{s}(\lambda+\xi,i\zeta/2)\,\mathfrak{s}(\lambda-\xi,i\zeta/2) \tag{2.4}$$

Lastly, we remind that $\rho(\lambda) = \dfrac{1}{\zeta\cosh(\pi\lambda/\zeta)}$ solves the Lieb equation:

$$\frac{\sin\zeta}{\pi\mathfrak{s}(\lambda,i\zeta/2)} = \rho(\lambda) + \int\limits_{\mathbb{R}}\mathrm{d}\mu\,K(\lambda-\mu)\,\rho(\mu) \ ; \qquad K(\lambda) = \frac{\sin 2\zeta}{2\pi\,\mathfrak{s}(\lambda,i\zeta)} \tag{2.5}$$

3 $\zeta = \pi/3$, an exact result.

When $\zeta = \pi/3$, the integrand of eq.(2.1) can be further simplified due to the duplication formula:

$$\sinh(3x) = 4\sinh(x)\sinh(x + i\pi/3)\sinh(x - i\pi/3) \quad . \tag{3.1}$$

Indeed, in this special point one recasts $\tau(m)$ as

$$\begin{aligned} \tau(m) &= \frac{(3/4)^{m(m+1)}}{\prod\limits_{i<j} \mathfrak{s}^2(\xi_i, \xi_j)} \int\limits_{\mathcal{C}(h)} \frac{\mathrm{d}^m\lambda}{m!\,(4\pi)^m} \prod_{k=1}^{m} \frac{\sinh(3\lambda_k)\sinh(2\lambda_k)\,\mathfrak{s}\left(\xi_k, i\hat{\xi}_-\right)}{\mathfrak{s}\left(\lambda_k, i\hat{\xi}_- + i\pi/6\right)} \\ &\quad \times \det_m\left[\frac{1}{\mathfrak{c}(\lambda_j, \xi_k)}\right]\det_m\left[\frac{1}{h(\lambda_j, \xi_k)}\right] \end{aligned} \tag{3.2}$$

A partial homogeneous limit has already been performed in the latter formula.

The symmetry of the integrand enables us to replace the Cauchy determinant $\det_m[1/\mathfrak{c}(\lambda_j, \xi_k)]$ by $m!$ times the product of its diagonal elements. This yields the sought separation of variables. In the case of a zero boundary field (*ie.* $\xi_- = i\pi/2$), the evaluation of the integrals leads to

$$\tau(m) = \left(\frac{3}{4}\right)^{m(m+1)} \frac{(-2)^m}{\prod\limits_{k=1}^{m}\sinh(2\xi_k)\sinh(\xi_k)} \frac{\det_m[g(\xi_j, \xi_k)]}{\prod\limits_{j<k}\mathfrak{s}^2(\xi_j, \xi_k)} \quad , \tag{3.3}$$

where

$$g(x, y) = \frac{\sinh(x+y)/2}{\sinh 3(x+y)/2} - \frac{\sinh(x-y)/2}{\sinh 3(x-y)/2} \tag{3.4}$$

It happens that it is possible to evaluate the homogeneous limit of (3.3). Indeed, we have the following equality of limits:

$$\lim_{\xi_k \to 0} \frac{\det_m[g(\xi_j, \xi_k)]}{\prod\limits_{k=1}^{m}\xi_k^2 \prod\limits_{i<j}^{m}\left(\xi_j^2 - \xi_k^2\right)^2} = \lim_{\substack{x_i \to 0 \\ y_i \to 0}} \frac{\det_m[g(x_j, y_k)]}{\prod\limits_{k=1}^{m} x_k\, y_k \prod\limits_{i<j}^{m}\left(x_j^2 - x_k^2\right)\left(y_j^2 - y_k^2\right)} \quad . \tag{3.5}$$

We can send the x's and the y's to zero in an arbitrary way when computing the limit in the RHS of (3.5). In particular, we can choose $x_i = \alpha\, i$ and $y_j = \alpha\,(m+j)$, and sent α to 0. Such a homogeneous limit is the $\alpha \to 0$ limit of $\det_m[U]$, where $U \in \mathcal{M}_m(\mathbb{C})$ is given by

$$U_{i,j} = \frac{\sinh\alpha(i+j+m)/2}{\sinh\beta(i+j+m)/2} - \frac{\sinh\alpha(j-i+m)/2}{\sinh\beta(j-i+m)/2} \qquad \beta = 3\alpha. \tag{3.6}$$

Actually Kuperberg [18] evaluated $\det_m[U]$ for all α and β:

$$\det_m[U] = \frac{\prod\limits_{i<j}^{2m} 2\sinh\beta(j-i)/2 \prod\limits_{\substack{i,j \\ 2|j}}^{2m+1} 2\sinh(\alpha + \beta(j-i))/2}{\prod\limits_{i,j}^{m} 4\sinh\beta(m+j-i)/2 \;\; \sinh\beta(m+j+i)/2} \quad . \tag{3.7}$$

Hence:

$$\lim_{\xi_k\to 0}\frac{\det_m\left[g\left(\xi_j,\xi_k\right)\right]}{\prod\limits_{k=1}^{m}\xi_k^2\prod\limits_{i<j}^{m}\left(\xi_j^2-\xi_k^2\right)^2}=\frac{1}{\prod\limits_{\substack{i<j\\1\leq}}^{m}\left(i^2-j^2\right)\prod\limits_{\substack{\leq i<j\\m+1}}^{2m}\left(i^2-j^2\right)}\lim_{\alpha\to 0}\alpha^{-2m^2}\frac{\det_m\left[U\right]}{2m!} \tag{3.8}$$

This allows to obtain a closed formula for the emptiness formation probability in the homogeneous limit:

$$\tau\left(m\right)=\frac{\left(3/4\right)^{m^2}}{\left(-2\right)^m\left(2m\right)!}\frac{\prod\limits_{i<j}^{2m}\left(j-i\right)\prod\limits_{\substack{i,j\\2|j}}^{2m+1}1+3\left(j-i\right)}{\prod\limits_{i,j}^{m}\left(m+j-i\right)\left(m+j+i\right)\prod\limits_{i<j}^{m}\left(j^2-i^2\right)\prod\limits_{\substack{\leq i<j\\m+1}}^{2m}\left(j^2-i^2\right)}\;. \tag{3.9}$$

Or in terms of factorials:

$$\tau\left(m\right)=3\frac{2^{m+1}}{4^{m^2}}\prod_{j=1}^{m}\left(3j-1\right)\prod_{k=1}^{m-1}\frac{\left(6k+3\right)!}{\left(2k+1\right)!}\left\{\frac{1}{\left(2m\right)!}\prod_{k=1}^{2m}\frac{k!}{\left(2m+k\right)!}\right\}^{\frac{1}{2}}\;. \tag{3.10}$$

It is now a matter of standard asymptotic analysis to extract the large m behavior of (3.10):

$$\tau\left(m\right)=\left(\frac{3}{4}\right)^{3m^2}\left(\frac{3\sqrt{3}}{4}\right)^{m}m^{\frac{1}{72}}\,C\;\;\left(1+\mathrm{o}\left(1\right)\right)\;\;, \tag{3.11}$$

where the constant C is expressed in terms of the Euler Gamma function $\Gamma\left(z\right)$, the Euler constant γ and of the Riemann Zeta function$\zeta\left(z\right)$.

$$\begin{aligned} C \;=\; & \frac{2^{\frac{1}{4}}\sqrt{2}}{\Gamma\left(2/3\right)}\exp\left(\frac{9}{2}\zeta'\left(-1\right)+\frac{5}{36}\left(1+\gamma\right)\right) \qquad (3.12)\\ & \exp\left\{2\int_0^{+\infty}\mathrm{d}t\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\left(\frac{2-\cosh t/3-\cosh t/6}{t^2}+\frac{2\sinh t/3+\sinh t/6}{6t}-\frac{5}{72}\right)\right\}\;. \end{aligned}$$

We stress that the qualitative behavior of $\tau\left(m\right)$ differs from the bulk one by the presence of an exponential factor e^{cm} in the asymptotics . This seems to be a general feature of the boundary model as a similar behavior appears at the free fermion point. Moreover the gaussian decay is twice faster than in the bulk.

4 The leading asymptotics of $\tau\left(m\right)$ at $\Delta=0$

The integral representation for the EFP also admits a separation of variables at $\Delta=0$ (*ie* $\zeta=\pi/2$). More precisely specifying eq.(2.1) to $\zeta=\pi/2$ we get

$$\tau(m) = \frac{2^{m(m+1)^2} \sin^{2m}\left(\hat{\xi}_-\right)}{(2\pi)^m} \det_m[T] \tag{4.1}$$

The entries of the matrix T are defined by an integral:

$$T_{jk} = \int\limits_{\mathcal{C}(h_-)} \frac{\sinh^2(2\lambda)}{\cosh^{k+l}(2\lambda)\left[\cosh(2\lambda) + \sin\left(2\hat{\xi}_-\right)\right]} \, \mathrm{d}\lambda \tag{4.2}$$

We perform the change variables $z = 2/\cosh(2\lambda) - 1$ in the above integral. Then $\tau(m)$ reads

$$\tau(m) = \frac{\det_m[R]}{(-2\pi h_-)^m} \qquad \text{where} \qquad R_{jk} = \mathrm{e}^{i\frac{\pi}{2}} \lim_{\epsilon \to 0^+} \int\limits_{\mathscr{C}_\epsilon} \frac{z^{j+k-2}}{z-a} \omega(z) \, \mathrm{d}z \quad , \tag{4.3}$$

and

$$\mathscr{C}_\epsilon = \begin{cases} \,]-1-i\epsilon\,; 1-i\epsilon\,[\bigcup \Gamma_+(a) & h_- \in \,]1\,;+\infty\,[\\ \,]-1-i\epsilon\,; 1-i\epsilon\,[& h_- \in \,]-\infty\,; 1\,[\end{cases} \tag{4.4}$$

$$\omega(z) = \sqrt{(z-1)(3+z)} \qquad \text{and} \qquad a = h_- + h_-^{-1} - 1 \equiv \frac{\chi + \chi^{-1}}{2}$$

In order to apply the results of Appendix A we recall the Wiener-Hopf factorization of $\sqrt{3+\cos\theta}$

$$\sqrt{3+\cos\theta} = \sqrt{\frac{u}{2}} \left(1 + u^{-1}\mathrm{e}^{i\theta}\right)^{\frac{1}{2}} \left(1 + u^{-1}\mathrm{e}^{-i\theta}\right)^{\frac{1}{2}} \, . \tag{4.5}$$

Specializing eq. (A.35) to the determinant representation for the EFP, we get

$$\tau(m) \sim C \frac{u^{\frac{m}{2}}}{2^{m^2}} \left(\frac{2}{m}\right)^{\frac{1}{8}} \begin{cases} (h_-\chi)^{-m} \sqrt{\dfrac{1+\chi^{-1}u^{-1}}{1+\chi^{-1}}}, & h_- < 1 \\ \\ \left(\dfrac{\chi}{h_-}\right)^m \sqrt{\dfrac{\chi^{-1}+u^{-1}}{1+\chi^{-1}}}, & h_- > 1 \end{cases} \, . \tag{4.6}$$

The constant C is expressed in terms of u and of the Barnes G-function

$$C = \left(\frac{1+u^{-1}}{1-u^{-1}}\right)^{\frac{1}{8}} \frac{\pi^{\frac{1}{4}} G(1/2)}{\sqrt{1+u^{-1}}} \tag{4.7}$$

One can see explicitly from the large m asymptotics of $\tau(m)$ the asymmetry induced by the boundary field. This asymmetry appears for $\mid h_- \mid > 1$. One has

$$\frac{\tau_{h_-}(m)}{\tau_{-h_-}(m)} = \sqrt{\frac{\left(u+\chi_+^{-1}\right)\left(1+\chi_-^{-1}\right)}{\left(u\chi_-^{-1}+1\right)\left(1+\chi_+^{-1}\right)}}\left(-\chi_+\chi_-\right)^m \qquad h_- > 1 \tag{4.8}$$

where

$$\frac{\chi_\pm + \chi_\pm^{-1}}{2} = \pm\left(h_- + h_-^{-1}\right) - 1 \; . \tag{4.9}$$

Finally, one infers from the asymptotics that $\tau(m) \underset{h_-\to-\infty}{\longrightarrow} 0$ for large m, as it should be. Indeed, in such a limit, the first spin is necessarily oriented upwards. Conversely, when $h_- \to +\infty$, $\tau(m)$ is not vanishing, also as expected.

5 The $m \to +\infty$ limit of $\ln(\tau(m))/m^2$

Using the saddle point approximation we obtain the leading asymptotics of $\ln(\tau(m))$. We find that the leading terms is gaussian and independent of the value of the boundary field h_-:

$$\ln(\tau(m)) = Km^2 + \mathrm{o}\left(m^2\right) \tag{5.1}$$

The constant K is twice bigger than the one appearing in the leading asymptotics of $\tau(m)$ of the periodic chain [14].
In order to implement the asymptotic analysis of $\ln\tau(m)$ we first recast $\tau(m)$ as

$$\tau(m) = \frac{|\sinh(\xi_-)|^{2m}}{(2\zeta)^{2m}}\left(\frac{\pi}{\sin(\zeta)}\right)^m\left(\frac{\pi}{\zeta}\right)^{2m^2}\mathcal{I}_m \tag{5.2}$$

$\mathcal{I}_m$ is an m-fold integral

$$\mathcal{I}_m = \frac{1}{2^m\, m!}\int\limits_{\mathcal{C}(h_-)} \mathrm{d}^m\lambda\ \mathrm{e}^{m^2\mathscr{S}(\{\lambda\}_1^m)}\mathrm{det}_m\left[M_{jk}\right] \tag{5.3}$$

$$\begin{aligned}\mathscr{S}\left(\{\lambda\}_1^m\right) &= \frac{1}{m^2}\sum_{k=1}^m \ln\left(\frac{\sinh^2\pi\lambda_k/\zeta}{\sinh(\lambda_k,\xi_-+i\zeta/2)}\right) + \frac{1}{m^2}\sum_{k<l}\ln\left(\frac{\mathfrak{s}^2\left(\pi\left(\lambda_k,\lambda_l\right)/\zeta\right)}{\mathfrak{s}\left(\lambda_{kl},i\zeta\right)\mathfrak{s}\left(\overline{\lambda}_{kl},i\zeta\right)}\right)\\ &\quad+\frac{2}{m}\sum_{k=1}^m\ln\left(\frac{\mathfrak{s}\left(\lambda_k,i\zeta/2\right)}{\cosh^2\pi\lambda_k/\zeta}\right) \quad ,\end{aligned} \tag{5.4}$$

and the matrix M is obtained by inverting the equation

$$\sum_{b=1}^m M_{ib}\Phi\left(\lambda_b,\xi_k\right) = \frac{\sinh(2\lambda_j)\sin\zeta}{\pi h\left(\lambda_j,\xi_k\right)} \quad . \tag{5.5}$$

We now reexpress $\mathcal{I}_m$ in a form more suited for an asymptotic analysis:

$$\mathcal{I}_m = \int\limits_{0<\lambda_1<\cdots<\lambda_m} \mathrm{d}^m\lambda \, \mathrm{e}^{m^2 \mathscr{S}(\{\lambda\}_1^m)} \det_m [M_{jk}] + \delta_{\hat{\xi}_-} \frac{\sinh^2 \pi\lambda_m/\zeta}{\sinh(2\lambda_m)} \int\limits_{\substack{0<\lambda_1<\cdots<\lambda_{m-1} \\ -i\lambda_m=\zeta/2+\hat{\xi}_-}} \mathrm{d}^{m-1}\lambda$$

$$\times \mathrm{e}^{m^2 \mathscr{S}(\{\lambda\}_1^{m-1})} \det_m \underbrace{\left[M_{jk} \frac{\mathfrak{s}(\lambda_m, i\zeta/2)\, \mathfrak{s}^2(\pi(\lambda_m,\lambda_j)/\zeta)}{\mathfrak{s}(\lambda_{jm}, i\zeta)\, \mathfrak{s}(\overline{\lambda}_{jm}, i\zeta) \cosh^4(\pi\lambda_m/\zeta)} \right]}_{\widetilde{M_{jk}}} \tag{5.6}$$

Here

$$\delta_{\hat{\xi}_-} = \begin{cases} 1 & \hat{\xi}_- \in \left] -\dfrac{\zeta}{2}; 0 \right[\\ 0 & \text{otherwise} \end{cases} . \tag{5.7}$$

Note that we have used the symmetry in the λ's as well as the $\lambda \to -\lambda$ invariance of the integrand in order to replace the integration over $\mathbb{R}^m$ by $2^m m!$ times the integration over the ordered domain $\{0 < \lambda_1 < \cdots < \lambda_m\}$. We also used the determinant structure and the symmetry of the integrand to evaluate the contribution of poles, if it exists.

The leading contributions to the above integrals will be equal to the integrand evaluated at the solutions of the saddle point equations. We assume that, in the $m \to +\infty$ limit, these solutions densify on $\mathbb{R}^+$ with a density $\sigma(\lambda)$. Then sums can be replaced by integrals according to the prescription

$$\frac{1}{m} \sum_{i=1}^{m} f(\lambda_i) \underset{m\to+\infty}{\longrightarrow} \int_0^{+\infty} \mathrm{d}\lambda \, \sigma(\lambda) \, f(\lambda) \tag{5.8}$$

It is worth noticing that in this limit, M_{jp} can be approximated by

$$M_{jp} = \delta_{jp} + \frac{K(\lambda_{jp}) - K(\overline{\lambda}_{jp})}{m\sigma(\lambda_j)} . \tag{5.9}$$

This can be seen by replacing the sums by integrals in (5.5) and using the integral equation (2.5). Then, Hadamard's inequality

$$|\det_m [a_{jk}]| \leq \left(\max_{j,k} (| a_{jk} |) \right)^m m^{\frac{m}{2}} \tag{5.10}$$

ensures that

$$\lim_{m\to+\infty} \frac{\ln \det_m [M_{j,k}]}{m^2} = \lim_{m\to+\infty} \frac{\ln \det_m \left[\widetilde{M}_{j,k}\right]}{m^2} = 0 . \tag{5.11}$$

As a consequence, the functions $\det_m [M]$ and $\det_m \left[\widetilde{M}\right]$ cannot contribute to the leading asymptotics of $\ln(\tau(m))$.

The density of saddle point roots $\sigma(\lambda)$ satisfies a singular integral equation. This equation is obtained by taking the large m limit in the saddle point equations $\dfrac{\partial \mathcal{S}(\{\lambda\})}{\partial \lambda_j} = 0$. This singular integral equation reads:

$$2\left(\frac{2\pi}{\zeta}\tanh\frac{\pi\lambda}{\zeta} - \coth(\lambda + i\zeta/2) - \coth(\lambda - i\zeta/2)\right) = \tag{5.12}$$
$$V.P.\int_{\mathbb{R}^+} \mathrm{d}\mu\sigma(\mu)\sum_{\epsilon=\pm}\frac{2\pi}{\zeta}\tanh\pi\frac{\lambda-\epsilon\mu}{\zeta} - \coth(\lambda-\epsilon\mu+i\zeta)\coth(\lambda-\epsilon\mu-i\zeta)$$

It is natural to extend the density into an even function on $\mathbb{R}$. This recasts the integral equation into a form very close to the integral equation appearing in the bulk model [14]:

$$\frac{2\pi}{\zeta}\tanh\frac{\pi\lambda}{\zeta} - \coth(\lambda + i\zeta/2) - \coth(\lambda - i\zeta/2) = \tag{5.13}$$
$$V.P.\int_{\mathbb{R}} \mathrm{d}\mu\sigma(\mu)\left\{\frac{2\pi}{\zeta}\tanh\pi\frac{\lambda-\mu}{\zeta} - \coth(\lambda-\mu+i\zeta)\coth(\lambda-\mu-i\zeta)\right\} .$$

Eq.(5.14) is solved by the Fourier transform. We find

$$\hat{\sigma}(k) = \frac{2\cosh k\zeta/2}{\cosh k\zeta} \qquad \text{hence} \qquad \sigma(\lambda) = \frac{\sqrt{2}\cosh\pi\lambda/2\zeta}{\zeta\cosh\pi\lambda/\zeta} \tag{5.14}$$

It is then immediate to evaluate the $m \to +\infty$ limit of $\mathscr{S}(\{\lambda\}_1^m)$ at the solutions of the saddle point equations:

$$\lim_{m\to+\infty}\mathscr{S}(\{\lambda\}_1^m) = \mathscr{S}^{(\infty)} = \int_{\mathbb{R}-i0}\frac{\mathrm{d}k}{k}\frac{\cosh^2(k\zeta/2)\ \sinh[k(\pi-\zeta)/2]}{\cosh(k\zeta)\ \sinh(k\zeta/2)\sinh((k\pi)/2)} . \tag{5.15}$$

The two multiple integrals appearing in (5.6) have thus the same gaussian decay. Thus,

$$\lim_{m\to\infty}\frac{\ln(\tau(m))}{m^2} = 2\ln\left(\frac{\pi}{\zeta}\right) + \mathscr{S}^{(\infty)} \tag{5.16}$$

In particular we recover the results at $\Delta = 0$ and $\Delta = 1/2$. As already announced we obtain twice the corresponding bulk constant. Unfortunately the saddle point method cannot catch the exponential behavior of the leading asymptotics of $\tau(m)$ and in particular the dependence on the boundary field.

6 Conclusion

We have obtained the leading asymptotic behavior of the EFP of the open XXZ spin-1/2 chain for some particular values of the couplings. The $\Delta = 1/2$ point seems to have, at least at $h_- = 0$, similar symmetry properties to the bulk model. Thus it should be possible to evaluate to the end the integrals appearing the generating function of σ^z correlators at $\Delta = 1/2$ in the open chain. As suggested by the saddle point analysis, the presence of the boundary increases the long range gaussian decay of $\tau(m)$ by roughly a square factor with respect to the bulk. It would be interesting to derive some formula for the exponential term e^{cm} appearing in the asymptotics of $\ln(\tau(m))$ in the massless regime. All the more that this terms should exhibit a dependency on the boundary field h_-.

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A Appendix

In this Appendix we derive the leading asymptotics of a modified Hankel determinant $detmR$. The entries of R reads

$$R_{ij} = \mathrm{e}^{i\pi\alpha} \lim_{\epsilon \to 0^+} \int_{\mathscr{C}_\epsilon} \mathrm{d}z \, \frac{z^{j+k-2}\omega(z)}{\prod_{p=1}^{s} (z - a_p)} \, . \tag{A.1}$$

The integration is carried over a segment and loops encircling some poles of the integrand:

$$\mathscr{C}_\epsilon = [\,-1 - i\epsilon\,; 1 - i\epsilon\,] \bigcup_{p \in E} \Gamma_+(a_p) \qquad \text{with} \qquad E \subset [\![\, 1\,;\, s\,]\!] \, . \tag{A.2}$$

In what follows, we assume $a_p \in \mathbb{C} \setminus]\,-1\,;1\,]$ and $t \notin E$ whenever $a_p \in]\,-\infty\,;-1\,]$. Moreover we restrict to weight functions of modified Jacobi type:

$$\omega(z) = (z-1)^\alpha (1+z)^\beta g(z) \quad .$$

We also assume $g(z)$ holomorphic in an open neighborhood of $[\,-1\,;1\,] \bigcup_{p=1}^{s} \{a_p\}$ and such that $g([\,-1\,;1\,]) \subset \mathbb{R}^+$.

A.1 Reduction to a simpler problem

We first remove the eventual contour integrals from most of the lines of R. For $j < m-s$, a linear combination of $L_j, \ldots, L_{j+s}$ allows to perform the replacement

$$z^{j+k} \to z^{j+k} \prod_{p=1}^{s} (1 - z/a_p) \tag{A.3}$$

in the integrand. For $j \in [\![m-s\,;\, m-2]\!]$ one can only replace

$$z^{j+k} \to z^{j+k} \prod_{p=j+2-m+s}^{s} (1 - z/a_p) \quad . \tag{A.4}$$

The above transformations lead to

$$\det_m [R] = \det_m \left[\widetilde{R}\right] \prod_{p=1}^{s} (-a_p)^{s+1-m-p} \quad , \tag{A.5}$$

where

$$\widetilde{R}_{jk} = \begin{cases} \int\limits_{-1}^{1} \mathrm{d}z\, z^{j+k-2}\, \overline{\omega}\,(z) & 1 \leq j \leq m-s \\ \mathrm{e}^{i\alpha\pi} \lim_{\epsilon\to 0^+} \int\limits_{\mathscr{C}_\epsilon} \mathrm{d}z \, \dfrac{z^{j+k-2}\, \omega\,(z)}{\prod\limits_{p=1}^{j-m+s} (z-a_p)} & m-s < j \leq m \end{cases} \tag{A.6}$$

and $\overline{\omega}\,(x) = \mathrm{e}^{i\pi\alpha} \omega\,(x - i0^+)$

The second step separates the "pole" part of the determinant from its pure Hankel part. Let $\Pi_k\,(z)$ be the monoic orthogonal polynomials on $[-1\,;1]$ with respect to the weight $\overline{\omega}\,(x)$. The reconstruction of these polynomials in the first $m-s$ lines as well as in all the columns makes the first $m-s$ lines diagonal. This procedure splits $\det_m \left[\widetilde{R}\right]$ into a product of two determinants. Namely,

$$\det_m \left[\widetilde{R}\right] = \det_{m-s} [H] \det_s \left[K^{(m)}\right] \tag{A.7}$$

Where

$$H_{jk} = \int_{-1}^{1} z^{k+j-2}\, \overline{\omega}\,(z)\, \mathrm{d}z \tag{A.8}$$

$$K^{(m)}_{jk} = \lim_{\epsilon\to 0^+} \int_{\mathscr{C}_\epsilon} \frac{z^{j+m-s-1} \Pi_{k+m-s-1}\,(z)}{\prod\limits_{p=1}^{j} (z-a_p)} \omega\,(z)\, \mathrm{d}z \tag{A.9}$$

A.2 The Hankel Determinant

The asymptotics of the Hankel determinant can be obtained *via* a procedure established by Basor and Ehrhardt. They built a mapping between Hankel and Toeplitz determinants [19]. This mapping allowed them to infer the large size behavior of the Hankel determinant from the known asymptotics of the Toeplitz determinant. We recall their result in

Proposition A.1 *Let T and H be respectively a Toeplitz and a Hankel matrix*

$$T_{kl} = \frac{1}{2\pi}\int_{-\pi}^{\pi} c(\theta)\, e^{-i(k-l)\theta} d\theta \tag{A.10}$$

$$H_{kl} = \int_{-1}^{1} \varpi(x)\, x^{k+l-2} dx \tag{A.11}$$

such that $\varpi \in L^1[-1\,;1]$ *and*

$$c(\theta) = isgn(\theta)\,\varpi(\cos\theta) \qquad -\pi < \theta < \pi\ , \tag{A.12}$$

then

$$det_m[H] = 2^{-m(m-1)}\pi^m\sqrt{det_{2m}[T]} \tag{A.13}$$

The function $c : \theta \mapsto i\mathrm{sgn}(\theta)\,\varpi(\cos\theta)$ is of degenerate Fischer-Hartwig type [1] [16] as it admits two maximal decompositions into the canonical Fischer-Hartwig product:

$$\begin{aligned} c(\theta) &= 2^{-\alpha-\beta}h(\cos\theta)\, w_{\pi,\beta,1/2}\left(\mathrm{e}^{i\theta}\right) w_{0,\alpha,-1/2}\left(\mathrm{e}^{i\theta}\right) \\ &= -2^{-\alpha-\beta}h(\cos\theta)\, w_{\pi,\beta,-1/2}\left(\mathrm{e}^{i\theta}\right) w_{0,\alpha,1/2}\left(\mathrm{e}^{i\theta}\right) \end{aligned}$$

The function $w_{\theta_r,\tau,\sigma}\left(\mathrm{e}^{i\theta}\right)$ reads

$$w_{\theta_r,\tau,\sigma}\left(\mathrm{e}^{i\theta}\right) = \left(2-2\cos\left(\theta-\theta_r\right)\right)^{\tau}\mathrm{e}^{i\sigma(\theta-\pi-\theta_r)} \quad \theta_r < \theta < \theta_r + 2\pi \tag{A.14}$$

The assumptions made on $g(z)$ guarantee the existence of a Wiener-Hopf decomposition:

$$g(\cos\theta) = C[g]\, g_+\left(\mathrm{e}^{i\theta}\right) g_+\left(\mathrm{e}^{-i\theta}\right) \quad , \tag{A.15}$$

where $C[g]$ and $g_+(t)$ satisfy the constraints:

$$C[g] \in \mathbb{R} \qquad\qquad g_+(t) = \exp\left\{\sum_{n=1}^{+\infty} t^n [g]_n\right\}\ ;\ \ [g]_n \in \mathbb{R} \tag{A.16}$$

[1]If α or $\beta = -1/2$ then $c(\theta)$ is not degenerate. In this case one should drop the $(-1)^m$ factor in (A.17). Moreover the result doesn't rely on a conjecture at these points as asymptotics of Toeplitz determinants with such Fischer-Hartwig symbols are known [16].

The leading order asymptotics of H thus follow from the generalized Fischer-Hartwig conjecture raised by Basor and Tracy [20]:

$$\det_m\left[T_{kl}\left(c\right)\right]=\left(\frac{C\left[g\right]}{2^{\alpha+\beta}}\right)^m\frac{E\left[g\right]f_{\alpha,\beta}\,m^{\alpha^2+\beta^2-\frac{1}{2}}}{4^{\alpha\beta}g_+^{2\beta}\left(-1\right)g_+^{2\alpha}\left(1\right)}\left(\frac{1+\left(-1\right)^m}{\sqrt{2}}+\mathrm{o}\left(1\right)\right)\quad,\tag{A.17}$$

where

$$f_{\alpha,\beta}=\prod_{t=\alpha,\beta}\frac{G\left(1+t+1/2\right)G\left(1+t-1/2\right)}{G\left(1+2t\right)}\qquad\text{and}\qquad E\left[g\right]=\mathrm{e}^{\sum_1^{+\infty}n[g]_n^2}\ .\tag{A.18}$$

Here G is the Barnes G function.

The asymptotic behavior of the Hankel determinant reads:

$$\det_{m-s}\left[H\right]=2^{-(m-s)^2}\left(\frac{2\pi C\left[g\right]}{2^{\alpha+\beta}}\right)^{m-s}\frac{m^{\left(\alpha^2+\beta^2\right)/2}\,2^{\frac{(\alpha-\beta)^2}{2}}}{m^{\frac{1}{4}}\,g_+^{\beta}\left(-1\right)g_+^{\alpha}\left(1\right)}\sqrt{E\left[h\right]\;f_{\alpha\,\beta}}\;\left(1+\mathrm{o}\left(1\right)\right)\tag{A.19}$$

A.3 Asymptotics of $\det_s\left[K^{(m)}\right]$

Computing the poles and taking the $\epsilon\to0^+$ limit yields

$$\begin{aligned}K_{jk}^{(m)}\ &=\ \int_{-1}^{1}\frac{z^{j+m-s-1}\Pi_{k+m-s-1}\left(z\right)}{\prod_{p=1}^{j}\left(z-a_p\right)}\varpi\left(z\right)\mathrm{d}z\\&\qquad+2i\pi\mathrm{e}^{i\alpha\pi}\sum_{t=1}^{j}\frac{a_t^{m-s+j-1}\Pi_{k+m-s-1}\left(a_t\right)}{\prod_{\substack{p=1\\\neq t}}^{j}a_{tp}}\omega\left(a_t\right)1_E\left(t\right)\ .\end{aligned}\tag{A.20}$$

Here 1_E stands for the characteristic function of E. The asymptotic analysis of (A.21) is based on the uniform asymptotic estimates for the monoic orthogonal polynomials with respect to the modified Jacobi weight ϖ [17]. Let K be compact in $\mathbb{C}\setminus\left[-1\,;1\right]$ and $\tilde{K}$ compact in $\left]\,0\,;\pi\,\right[$. Then :

$$\Pi_n\left(\cos\theta\right)\ =\ \frac{D_\infty2^{1/2-n}}{\sqrt{\varpi\left(\cos\theta\right)\sin\left(\theta\right)}}\cos\left\{\left(n+\frac{\alpha+\beta+1}{2}\right)\theta+\Psi\left(\theta\right)\right\}\epsilon_n\tag{A.21}$$

$$\Pi_n\left(z\right)\ =\ \frac{D_\infty}{D\left(\overline{\omega},z\right)}\frac{\left(\chi/2\right)^n}{\left(1-\chi^{-2}\right)^{1/2}}\,\epsilon_n'\qquad z=\frac{\chi+\chi^{-1}}{2}\ ,\mid\chi\mid>1\ .\tag{A.22}$$

One has $\epsilon_n, \epsilon'_n = \left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right)$ uniformly in $\theta \in \tilde{K}$ and respectively $z \in K$. Moreover have introduced the functions:

$$\Psi(\theta) = \frac{i}{2} \ln\left(\frac{g_+\left(\mathrm{e}^{-i\theta}\right)}{g_+\left(\mathrm{e}^{i\theta}\right)}\right) - \frac{\pi}{2}\left(\frac{1}{2} + \alpha\right) \tag{A.23}$$

$$D(\overline{\omega}, z) = \sqrt{\frac{G[g]}{2^{\alpha+\beta}}}\, g_+\left(\chi^{-1}\right)\left(1 - \chi^{-1}\right)^{\alpha}\left(1 + \chi^{-1}\right)^{\beta} \tag{A.24}$$

$$D_\infty = \lim_{\Re z \to +\infty} D(\overline{\omega}, z) = \sqrt{\frac{G[g]}{2^{\alpha+\beta}}} \tag{A.25}$$

Recall that we have decomposed z into $z = \left(\chi + \chi^{-1}\right)/2$, $\mid \chi \mid > 1$ and Wiener-Hopf factorized g. It follows immediately from (A.22) that the leading order of the "pole part" in (A.21) reads

$$\frac{a_t^{m-s+j-1} \Pi_{k+m-s-1}(a_t)}{\prod\limits_{\substack{p=1\\ \neq t}}^{j} a_{tp}} \omega(a_t) = \frac{C[g]}{2^{\alpha+\beta}} \frac{a_t^{m-s+j-1} (\chi_t/2)^{m-s+k-1/2}}{\sqrt{\left(1 - \chi_t^{-2}\right)}}$$
$$\times\, g_+(\chi_t)\; \frac{(\chi_t - 1)^{\alpha} (\chi_t + 1)^{\beta}}{\prod\limits_{\substack{p=1\\ \neq t}}^{j} a_{tp}} \epsilon'_n \tag{A.26}$$

We have used the identity

$$\frac{(\chi_t - 1)^{\alpha} (\chi_t + 1)^{\beta}}{2^{\alpha+\beta}} = \frac{(a_t - 1)^{\alpha} (a_t + 1)^{\beta}}{\left(1 - \chi_t^{-1}\right)^{\alpha}\left(1 + \chi_t^{-1}\right)^{\beta}} \tag{A.27}$$

We now derive the asymptotics of the integral term in (A.21). This amounts to the study of the asymptotics $n, q \to +\infty$, $n - q + j \geq 1$, of the sequence

$$\mathcal{I}_{q,n} = \int_{-1}^{1} \frac{z^q\, \Pi_n(z)}{\prod\limits_{p=1}^{j} (z - a_p)} \overline{\omega}(z)\, \mathrm{d}z \tag{A.28}$$

Applying the formula for the asymptotics of the orthogonal polynomials (A.21) we get:

$$\mathcal{I}_{qn} = \frac{D_\infty}{2^{n+\frac{1}{2}}} \int_0^{\pi} \mathrm{d}\theta\, \frac{\cos^q\theta \sqrt{g(\cos\theta)}}{\prod_{p=1}^{j} (\cos\theta - a_p)} (1 + \cos\theta)^{\frac{\beta}{2}+\frac{1}{4}} (1 - \cos\theta)^{\frac{\alpha}{2}+\frac{1}{4}}$$
$$\times \left(\mathrm{e}^{i\left(n + \frac{1+\alpha+\beta}{2}\right)\theta + i\Psi(\theta)} + \mathrm{e}^{-i\left(n + \frac{1+\alpha+\beta}{2}\right)\theta - i\Psi(\theta)}\right) \epsilon_n \tag{A.29}$$

We recast the integral as one over $]-\pi\,;\pi\,[$ and then, using the analycity of the integrand, we shift the contour to $\theta\in\,]-\pi+i0^+\,;\pi+i0^+\,[$. Since

$$(\cos\theta-1)^\gamma=\left\{\begin{array}{ll}(1-\cos\theta)^\gamma\,\mathrm{e}^{-i\frac{\gamma\pi}{2}} & \theta\in\,]\,i0^+\,;\pi+i0^+\,[\\ (1-\cos\theta)^\gamma\,\mathrm{e}^{i\frac{\gamma\pi}{2}} & \theta\in\,]-\pi+i0^+\,;i0^+\,[\end{array}\right. \tag{A.30}$$

we can absorb the factors $\mathrm{e}^{\mp i\frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)}$. Moreover $2^\gamma\,(\cos\theta\pm1)^\gamma=\mathrm{e}^{-i\gamma}\left(1+\pm\mathrm{e}^{i\theta}\right)^\gamma$ $\theta\in\,]-\pi+i0^+\,;\pi+i0^+\,[$. Finally, we obtain

$$\mathcal{I}_{q,n}=\frac{C\,[g]}{2^{n+\alpha+\beta+1}}\int\limits_{-\pi+i0^+}^{\pi+i0^+}\frac{\left(1+\mathrm{e}^{i\theta}\right)^{\beta+\frac{1}{2}}\left(1-\mathrm{e}^{i\theta}\right)^{\alpha+\frac{1}{2}}\cos^q\theta}{\prod\limits_{p=1}^{j}(\cos\theta-a_p)}\,\mathrm{e}^{in\theta}g_+\left(\mathrm{e}^{i\theta}\right)\epsilon_n\,\mathrm{d}\theta \tag{A.31}$$

The above integral can be interpreted as a contour integral over $\Gamma\,(0,1-0^+)$.

$$\mathcal{I}_{q,n}=\frac{C\,[g]}{2^{n+\alpha+\beta+1}}\oint\limits_{\Gamma(0,1-0^+)}\frac{\mathrm{d}\zeta}{i\zeta}\frac{(1+\zeta)^{\beta+1/2}\,(1-\zeta)^{\alpha+1/2}}{\prod\limits_{p=1}^{j}\left(\zeta+\zeta^{-1}-2a_p\right)/2}\,g_+\,(\zeta)\,\zeta^n\left(\frac{\zeta^{-1}+\zeta}{2}\right)^q\,\epsilon_n$$

The integrand has no pole at $\zeta=0$; indeed it behaves as $\zeta^{j-1+n-p}$, $\zeta\to0$, and we have imposed $n+j-p-1\geq0$. The only poles inside of the contour are in the points $\zeta=\chi_t^{-1}$, $t\in[\![\,1\,;\,j\,]\!]$ where $2a_t=\chi_t+\chi_t^{-1}$ and $\mid\chi_t\mid>1$. So

$$\mathcal{I}_{q,n}=\frac{2\pi C\,[g]}{2^{n+\alpha+\beta}}\sum_{t=1}^{j}a_t^q\,g_+\left(\chi_t^{-1}\right)\frac{\left(1-\chi_t^{-1}\right)^{\alpha+\frac{1}{2}}\left(1+\chi_t^{-1}\right)^{\beta+\frac{1}{2}}}{\chi_t^n\left(\chi_t^{-1}-\chi_t\right)\prod\limits_{\substack{p=1\\ \neq t}}^{j}a_{tp}}\epsilon_n\quad. \tag{A.32}$$

Thus $K_{jk}^{(m)}$ has the asymptotic behavior

$$\begin{aligned}K_{jk}^{(m)}\;&=\;\frac{2\pi C\,[g]}{2^{m-s+k+\frac{1}{2}}}\sum_{t=1}^{j}\Big\{i\mathrm{e}^{i\pi\alpha}g_+\,(\chi_t)\,\chi_t^{m-s+k-1}\,(\chi_t-1)^\alpha\,(1+\chi_t)^\beta\;1_E\,(t)\\ &\qquad-g_+\left(\chi_t^{-1}\right)\chi_t^{s-m-k}\left(1-\chi_t^{-1}\right)^\alpha\left(1+\chi_t^{-1}\right)^\beta\Big\}\frac{a_t^{m-s+j-1}\,(1+\mathrm{o}\,(1))}{\sqrt{1-\chi_t^{-2}}\prod\limits_{\substack{p=1\\ \neq t}}^{j}a_{tp}}\end{aligned}$$

One can simplify the asymptotics of $\det_s\left[K^{(m)}\right]$ by performing the linear combination of lines:

$$L_j \leftarrow L_j - \sum_{t=1}^{j-1} \frac{a_t^{j-1+t}}{\prod\limits_{p=t+1}^{j} a_{tp}} L_t \qquad j = 2 \dots s \tag{A.33}$$

Then

$$\begin{aligned} \det_s \left[K^{(m)}\right] &= \frac{2^{\frac{s(s+1)}{2}}}{2^{ms}} \left(\frac{2\pi C[g]}{2^{\alpha+\beta}}\right)^s \frac{1}{\prod\limits_{k<j} a_{jk}} \prod_{t=1}^{s} \frac{a_t^{m-s+t-1}}{\sqrt{1-\chi_t^{-2}}} \left(1+\mathrm{o}(1)\right) \\ &\times \det_s \Big[i\mathrm{e}^{i\pi\alpha} g_+(\chi_t)\, \chi_t^{m-s+k-1} (\chi_t - 1)^\alpha (1+\chi_t)^\beta\, 1_E(t) \\ &\qquad - g_+\left(\chi_t^{-1}\right) \chi_t^{s-m-k} \left(1-\chi_t^{-1}\right)^\alpha \left(1+\chi_t^{-1}\right)^\beta \Big] \end{aligned} \tag{A.34}$$

Note that the terms coming from the integration over $]-1;1[$ are exponentially subleading with respect to the ones coming from the residues. Nevertheless we keep them in the final formula as we could have $t \notin E$. For instance, in the limit $E = \emptyset$ we recover the expected formula (A.19) for the asymptotics of Hankel determinants with a weight $\overline{\omega} / \prod\limits_{p=1}^{j} (z - a_p)$.

Summing up all the results, we get the formula for the leading asymptotics of $\det_m [R]$

$$\begin{aligned} \det_m [R] &= 2^{-m^2} \left(\frac{2^{s+1}\pi C[g]}{(-1)^s 2^{\alpha+\beta}}\right)^m \frac{m^{\frac{\alpha^2+\beta^2-1/2}{2}} 2^{\frac{(\alpha-\beta)^2}{2}}}{g_+^\beta(-1)\, g_+^\alpha(1) \prod\limits_{p>k}^{s} 2a_{kp}} \sqrt{\frac{f_{\alpha,\beta} E[g]}{\prod\limits_{t=1}^{s}\left(1-\chi_t^{-2}\right)}} \\ &\det_s \Big[g_+\left(\chi_t^{-1}\right) \chi_t^{s-m-k} \left(1-\chi_t^{-1}\right)^\alpha \left(1+\chi_t^{-1}\right)^\beta \\ &\quad - i\mathrm{e}^{i\pi\alpha} g_+(\chi_t)\, \chi_t^{m-s+k-1} (\chi_t-1)^\alpha (1+\chi_t)^\beta\, 1_E(t) \Big] \left(1+\mathrm{o}(1)\right) \end{aligned} \tag{A.35}$$

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